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**LETTER TO THE EDITOR**

# The synthetic method in radiative transfer theory

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**Abstract.** The utility of the synthetic method for performing non-LTE line radiation transfer calculations is investigated. The stability and convergence of the method are determined for an idealised problem. This analysis is compared to numerical calculations. Regimes where the method is likely to be useful are identified.

The standard methods of solving the radiation transfer equation for a spectral line (Mihalas 1978) require that large matrix equations be solved. In these approaches one solves for the radiation field at all frequencies of interest simultaneously. This enables some account of the effect of the radiation field on the source function (the ratio of emissivity to opacity) to be included implicitly.

An alternative approach, $\Lambda$ iteration, is to take the emissivity and opacity as given. The transfer equation may then be solved one frequency at a time. This is relatively cheap computationally. The emissivity and opacity must then be recalculated and the whole process must be iterated.

The relative expense of solving the radiative transfer equation by $\Lambda$ iteration or the more standard methods is highly problem dependent. In problems encountered in stellar atmospheres the number of photon scatters before escape from the atmosphere or collisional destruction may be very large. In such cases $\Lambda$ iteration may take very many iterations to converge. However in most laboratory sources very different conditions prevail. Photons may scatter only 10–1000 times before escape or destruction. In such cases $\Lambda$ iteration can be useful. However, some of the harder problems in modelling laboratory sources may require hundreds or thousands of iterations. Schemes which allow acceleration of the $\Lambda$ iteration are thus of great interest.

Two acceleration schemes have been applied to radiative transfer problems: the synthetic method (Cannon 1973, Scharmer 1981) and the Chebyshev semi-iterative scheme (Bond 1983). The convergence of the latter, for model problems, is straightforward to analyse, requiring only the knowledge of the spectral radius of the $\Lambda$ iteration operator. In this paper we analyse the stability of the synthetic scheme for radiative transfer applications. The method developed by Reed (1973) is used in the analysis. Unlike the case of one group transport (Alcouffe 1977) it is not possible to choose a low-order diffusion-like operator which is always stable. The stability and convergence properties of the synthetic method are calculated and compared with numerical calculations.

 

Consider the problem of radiative transfer in a spectral line. In slab geometry, and with the assumption of complete redistribution, we can write the transfer equation as

$$(\mathrm{d}/\mathrm{d}\tau)\phi_\nu^{-1}(\mathrm{d}/\mathrm{d}\tau)(fJ_\nu)+\phi_\nu J_\nu=\phi_\nu S_\mathrm{L} \tag{1}$$

where $\phi_\nu$ is the line absorption profile and $\tau$ the line centre optical depth;

$$J_\nu=\int_0^{+1} u\,\mathrm{d}\mu \qquad \text{where } u=\tfrac{1}{2}(I_+ + I_-)$$

$$K_\nu=\int_0^{+1} u\mu^2\,\mathrm{d}\mu$$

$$f_\nu = K_\nu/J_\nu.$$

The line source function, $S_\mathrm{L}$, is taken to have the form

$$S_\mathrm{L}=\varepsilon+(1-\varepsilon)\int \mathrm{d}\nu\,\phi_\nu J_\nu. \tag{2}$$

Equation (1) must be supplemented by boundary conditions which give the incident flux into the region being modelled.

In the analysis of the stability properties of the synthetic and $\Lambda$ iteration schemes we make the following simplifying assumptions: the Eddington factors, $f_\nu$, all have the same value; $\phi_\nu$ is not space dependent; in the numerical solution of the transfer equation the mesh spacing, $\Delta\tau$, is constant.

Next we use the methods developed by Reed (1973) to investigate the convergence/stability of the $\Lambda$ and synthetic iteration schemes.

The $\Lambda$ iteration consists of solving equation (1) for an assumed value of $S_\mathrm{L}$, recalculating $S_\mathrm{L}$ from equation (2) and iterating. The differenced version of equation (1) may be written

$$\frac{1}{\phi_\nu}\frac{f}{(\Delta\tau)^2}DJ^{n+1}+\phi_\nu J^{n+1}=\phi_\nu[\varepsilon+(1-\varepsilon)\{J\}^n] \tag{3}$$

where the superscript denotes the number of iterations, we have defined

$$-D\chi^n \equiv \chi^n_{L-1}-2\chi^n_L+\chi^n_{L+1} \qquad L=\text{mesh index}$$

and

$$\{\chi_\nu\}\equiv\int_{-\infty}^{+\infty}\phi_\nu\chi_\nu\,\mathrm{d}\nu.$$

The eigenvalues and eigenvectors of the operator, $D$, have been found by Reed (1973). They are:

$$\lambda_k=2(1-\cos\pi/k) \qquad \text{and} \qquad \Phi_k=\cos(L\pi/k) \qquad k=1,2\ldots$$

respectively.

We wish to find the amplification factors for error vectors. Neglecting the inhomogeneous term in equation (3) and combining this with equation (2) gives:

$$g_k=(1-\varepsilon)[1+(\lambda_k f/\phi_\nu^2\Delta\tau^2)]^{-1}. \tag{4}$$

Clearly $g_k$ is always less than unity, so the scheme is unconditionally stable for the

model problem. In a finite atmosphere, optical depth $\tau$, the slowest converging mode will be that with the largest value of $k$. This gives

$$g_{\max} \simeq (1-\varepsilon)[1+(f\pi^2/\phi_\nu^2\tau^2)]^{-1}.$$

From this it can be seen that $\Lambda$ iteration is effective if $\varepsilon$ is large or $\tau$ small, a well known result.

In the synthetic method equation (3) is solved for an intermediate value of the flux $\tilde{J}_\nu^{n+1}$. The following equation is then solved for the new flux $J_\nu^{n+1}$:

$$-d\frac{d^2}{d\tau^2}\{J\}^{n+1}+\varepsilon\{J\}^{n+1}=\varepsilon-d\frac{d^2}{d\tau^2}\{\tilde{J}\}^{n+1}-\operatorname{div}(\widetilde{\text{flux}})^{n+1}. \tag{5}$$

The flux is given by

$$\text{flux}=-\int d\nu\frac{1}{\phi_\nu}\frac{d}{d\tau}fJ_\nu.$$

Upon convergence $\{\tilde{J}\}^{n+1}=\{J\}^{n+1}$ and equation (5) converges to the photon balance equation. The parameter, $d$, is taken to be independent of position in the following analysis.

To find the amplification factors of the error vectors associated with the synthetic method we consider the homogeneous versions of equation (3), the differenced version of equation (5) and equation (2). This gives the amplification factors as:

$$g_k=[(d\lambda_k/(\Delta\tau)^2)(1-\varepsilon)I_1-(1-\varepsilon)I_2]/[(d\lambda_k/(\Delta\tau)^2)+\varepsilon]$$

where

$$I_1=[1+(\lambda_k f/\phi_\nu^2(\Delta\tau)^2)]^{-1}$$

and

$$I_2=\phi_\nu^2(\lambda_k f/(\Delta\tau)^2)/[1+\lambda_k f/\phi_\nu^2(\Delta\tau)^2].$$

This scheme should be considered successful if it gives a maximum value of the amplification vector much less than that given by the $\Lambda$ iteration, equation (4).

The maximum value of $g_k$ has been studied as a function of $d$. Clearly as $d\to\infty$ this scheme gives the same results as $\Lambda$ iteration. On the other hand too small a value of $d$ may result in the scheme becoming unstable. Figure 1 shows the maximum value of $g_k$ as a function of $d$ for atmospheres with different values of optical depth, $\tau$, and $\varepsilon$. Both Doppler and Lorentz line profiles have been used.

These results have been compared with code calculations in which the same values of $\varepsilon$, $\phi_\nu$ and $\tau$ were used. In these calculations $f$ and $\Delta\tau$ were position dependent. Good agreement ($\pm 5\%$) on the value of $d$ for which the scheme became unstable was found.

We have investigated the application of the synthetic method to radiative transfer calculations in which complete redistribution has been assumed. It is possible to achieve considerably faster convergence than is possible with $\Lambda$ iteration. The value of the parameter, $d$, for which the best convergence is achieved is, however, very dependent on the line shape and the optical depth. Instability is possible if too low a value of $d$ is used. Thus the application of the synthetic method to radiative transfer problems poses greater difficulties than its application to one group neutron transport

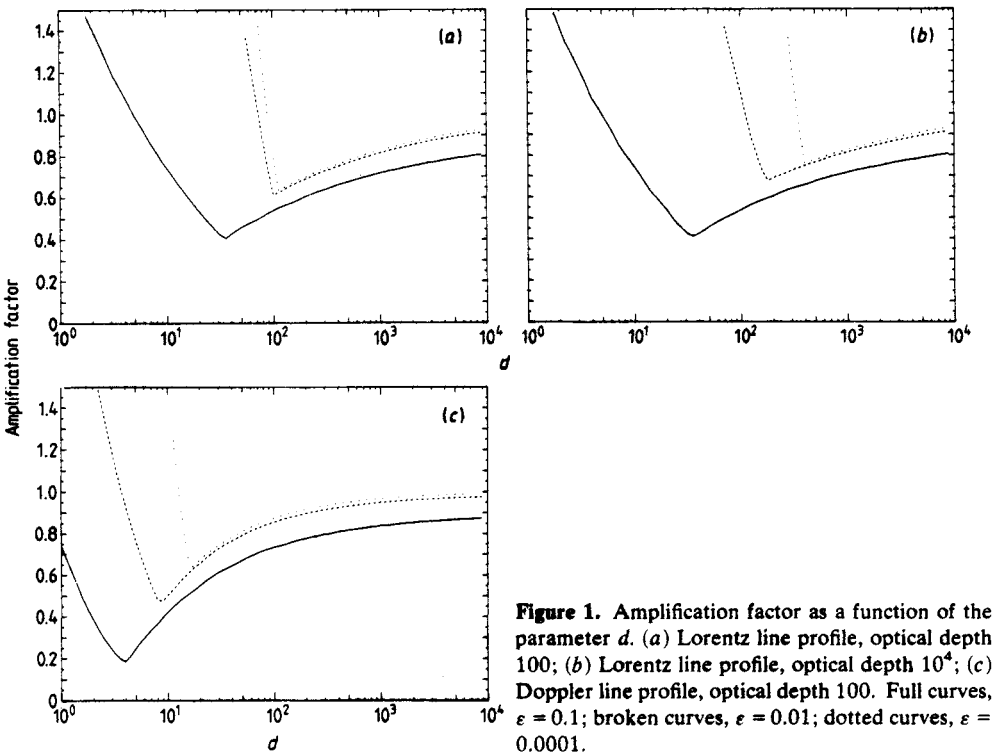


**Figure 1.** Amplification factor as a function of the parameter $d$. (*a*) Lorentz line profile, optical depth 100; (*b*) Lorentz line profile, optical depth $10^4$; (*c*) Doppler line profile, optical depth 100. Full curves, $\varepsilon = 0.1$; broken curves, $\varepsilon = 0.01$; dotted curves, $\varepsilon = 0.0001$.

problems. Since the eigenvector giving rise to the largest amplification factor may change as a function of $d$ an iteration scheme in which different values of $d$ are used at each iteration may have some merit.

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